Spectral equivalences and symmetry breaking in integrable $SU_q(N)$ spin chains with boundaries

A. Nichols[1]

Physikalisches Institut der Universität Bonn,
Nussallee 12, 53115 Bonn, Germany.

Abstract

We consider the $SU_q(N)$ invariant spin chain with diagonal and non-diagonal integrable boundary terms.

The algebraic study of spin chains with different types of boundary terms is used to motivate a set of spectral equivalences between integrable chains with purely diagonal boundary terms and ones with an arbitrary non-diagonal term at one end. For each choice of diagonal boundary terms there is an isospectral one-boundary problem and vice-versa.

The quantum group $SU_q(N)$ symmetry is broken by the presence of a non-diagonal boundary term however one can use the spectral equivalence with the diagonal chain to easily understand the residual symmetries of the system.

[1]nichols@th.physik.uni-bonn.de

1 Introduction

In a previous paper [1] we proved a rather surprising spectral equivalence between an XXZ chain with arbitrary left boundary term and the same XXZ chain with purely diagonal boundary terms. In this paper we shall show that this phenomenon extends to all $SU_q(N)$ spin chains where, due to the many different choices of diagonal boundary terms, the structure is much richer.

The XXZ chain with diagonal boundary conditions has been well studied using the Bethe Ansatz [2,3] due to the existence of a conserved charge S^z. For a particular choice of diagonal boundary terms the chain has an $SU_q(2)$ quantum group symmetry [4]. An alternative understanding of the properties of the quantum group invariant chain comes from noting that it can be written in terms of the generators of the Temperley-Lieb (TL) algebra [5, 6]. This algebra depends on a single parameter.

The addition of an arbitrary left boundary operator can be understood algebraically in terms of the one-boundary Temperley-Lieb algebra (1BTL) [7–10]. Although the integrable chain now involves three parameters the 1BTL contains only two of these. The presence of the boundary term breaks the $SU_q(2)$ symmetry however a remnant survives [11–14]. This element commutes with all elements of the 1BTL and at generic points of the algebra has the structure and degeneracies of a $U(1)$ charge whilst for exceptional points it becomes indecomposable [1].

A consequence of the spectral equivalence between the one-boundary and diagonal chain is that there must exist a representation of the 1BTL within the diagonal chain. We would like to stress that this is extremely surprising as conventionally the diagonal chain is considered to require *two* boundary terms. By direct calculation at a low number of sites one can find the explicit form for the generators of the 1BTL in the diagonal chain (see Appendix C of [1]). One finds that, although both the bulk and boundary generators commute with the diagonal charge S^z, they all become non-local. In a two site example it was realized in [1] that these non-local expressions could be brought to a form in which the bulk generators are given by their standard local expressions and the boundary generator, which still commutes with S^z, is non-local. In this paper we shall describe how to understand, and generalize, this canonical diagonal representation.

For $SU_q(N)$ spin chains, in contrast to the $SU_q(2)$ case, there are many different choices of integrable diagonal boundary terms [15]. In this paper we shall use an algebraic approach [16–18] to construct solutions to the reflection equation. The main advantage of such an approach, over more direct approaches [19–21], is that it gives a much more transparent way of understanding the structure of possible integrable boundary terms. Once one completes the 'Baxterization' procedure [22] then any representation of the algebra can be used to construct a solution to the Yang-Baxter and reflection equations. We shall show that each of the possible integrable diagonal boundary terms is related to a different one-boundary problem and moreover that all diagonal boundary conditions can be related in this way. A considerable amount

of insight into the existence of spectral equivalences can be gained from examining the canonical diagonal representation in which, as in the $SU_q(2)$ case, all the bulk generators keep their standard forms but the boundary generator becomes non-local. This is not quite the representation that give rise to the diagonal chain however it does encode the algebraic properties of the diagonal representation and allows one to see both the structure of conserved charges in one-boundary systems and the possibility of spectral equivalences. This shows that the results in [1] are the first in a large class of equivalences that can be found between one-boundary and diagonal systems.

In section 2 we review the spectral equivalence between the diagonal and one-boundary Temperley-Lieb chains found in [1]. The results from a low number of sites motivate the discussion of the non-local diagonal representations of 1BTL. There is a canonical diagonal representation in which only the boundary generator is non-local. In order to understand, even in the 1BTL case, this representation it is essential to introduce the braid group structure. In section 3 we review the appearance of braid, Hecke, and boundary algebras from the standpoint of integrability. In most of this paper we shall be concerned with the braid limits in which spectral parameters disappear. The full solution, with spectral parameters, will be required in order to discuss the integrable chains and can be obtained by a process of Baxterization (see appendix B). In section 4 we discuss the $SU_q(N)$ invariant spin chain and the possible integrable diagonal and non-diagonal boundary terms. In section 5 we give an expression for a boundary generator formed non-locally from the diagonal generators - the canonical diagonal representation. The existence of such a representation motivates the construction of spectral equivalences between integrable one-boundary and diagonal chains. In section 6 we use these results to understand how the $SU_q(N)$ symmetry is broken in the presence of an integrable boundary term. Finally we present our conclusions and some outstanding questions. In appendix A we give a diagonal representation of 1BTL in the link pattern basis. This is the representation which was used in [23].

2 $SU(2)$ spectral equivalences

In this section we shall review the results of [1], concerning the XXZ model with boundaries, which will be important for this paper.

We begin with the integrable $SU_q(2)$ quantum group Hamiltonian [4]:

$$H^{qg} = -\frac{1}{2}\left\{\sum_{i=1}^{L-1}\left(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y-\cos\gamma\sigma_i^z\sigma_{i+1}^z+\cos\gamma\right)+i\sin\gamma\left(\sigma_1^z-\sigma_L^z\right)\right\} \quad (2.1)$$

The central result, proved in [1] using the Bethe ansatz, is an exact spectral equivalence between the integrable $SU_q(2)$ chain with arbitrary non-diagonal boundary term at one end:

$$H^{nd} = \frac{\sin\gamma}{\cos\omega+\cos\delta}\left(i\cos\omega\sigma_1^z+\sigma_1^x-\sin\omega\right)$$

$$-\frac{1}{2}\left\{\sum_{i=1}^{L-1}\left(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y-\cos\gamma\sigma_i^z\sigma_{i+1}^z+\cos\gamma\right)+i\sin\gamma\left(\sigma_1^z-\sigma_L^z\right)\right\} \quad (2.2)$$

and the integrable chain with purely diagonal boundary conditions:

$$\begin{aligned} H^d &= -\frac{1}{2}\left\{\sum_{i=1}^{L-1}\left(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y-\cos\gamma\sigma_i^z\sigma_{i+1}^z+\cos\gamma\right)\right. \\ &\left.+\sin\gamma\left[\tan\left(\frac{\omega+\delta}{2}\right)\sigma_1^z+\tan\left(\frac{\omega-\delta}{2}\right)\sigma_L^z+\frac{2\sin\omega}{\cos\omega+\cos\delta}\right]\right\} \end{aligned} \quad (2.3)$$

For the $SU(2)$ case all choices of boundary terms are in fact integrable. When we generalize this spectral equivalence to $SU(N)$ with $N>2$ we shall see that integrability plays a much more restrictive role.

The $SU_q(2)$ quantum group invariant chain (2.1) can be written in terms of the Temperley-Lieb algebra:

$$H^{qg} = -\sum_{i=1}^{L-1} e_i \quad (2.4)$$

where the generators e_i $(i=1,\cdots,L-1)$ are given by:

$$e_i=\frac{1}{2}\left\{\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y-\cos\gamma\sigma_i^z\sigma_{i+1}^z+\cos\gamma+i\sin\gamma\left(\sigma_i^z-\sigma_{i+1}^z\right)\right\} \quad (2.5)$$

and obey the relations:

$$\begin{aligned} e_ie_{i\pm1}e_i &= e_i \\ e_ie_j &= e_je_i \quad |i-j|>1 \\ e_i^2 &= \left(q+q^{-1}\right)e_i \end{aligned} \quad (2.6)$$

with $q=e^{i\gamma}$. The addition of an integrable boundary term can be expressed in terms of the 1BTL algebra [7–10]. This is an extension of TL involving a boundary operator e_0.

$$\begin{aligned} e_1e_0e_1 &= e_1 \\ e_0^2 &= \frac{\sin\omega}{\sin(\omega+\gamma)}e_0 \\ e_0e_i &= e_ie_0 \quad i>1 \end{aligned} \quad (2.7)$$

With the bulk e_i defined in (2.5) one can verify that the expression [7]:

$$\begin{aligned} e_0 &= -\frac{1}{2}\frac{1}{\sin(\omega+\gamma)}\left(i\cos\omega\sigma_1^z+\sigma_1^x-\sin\omega\right) \\ &= -\frac{1}{2}\frac{1}{\sin(\omega+\gamma)}\begin{pmatrix} ie^{i\omega} & 1 \\ 1 & -ie^{-i\omega}\end{pmatrix}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1} \end{aligned} \quad (2.8)$$

obeys the 1BTL (2.7). The integrable chain is given by:

$$H^{nd} = -ae_0 - \sum_{i=1}^{L-1} e_i \tag{2.9}$$

Now with the parameterization:

$$a = \frac{2\sin\gamma\sin(\omega+\gamma)}{\cos\omega+\cos\delta} \tag{2.10}$$

we obtain the non-diagonal Hamiltonian (2.2).

Right boundary terms can also be written using an extension known as the two-boundary Temperley-Lieb (2BTL) algebra [24, 25] where, in addition to the 1BTL relations (2.7), we also have:

$$\begin{aligned} e_L e_{L-1} e_L &= e_L \\ e_L^2 &= \frac{\sin\omega'}{\sin(\omega'+\gamma)} e_L \\ e_L e_i &= e_i e_L \quad i < L-1 \end{aligned} \tag{2.11}$$

In contrast to the TL and 1BTL algebras, the 2BTL is infinite dimensional. The spin chain lives in a particular finite dimensional quotient [23, 25]. Here we shall only consider two-boundary cases in which both ends are diagonal. This is an important limiting case of the above, in which we take $w = w' = i\infty$, and the left and right diagonal terms are given by:

$$e_0^d = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{1} \otimes ... \mathbf{1} \qquad e_L^d = \mathbf{1} \otimes \mathbf{1} \otimes ... \otimes \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \tag{2.12}$$

Therefore the diagonal Hamiltonian (2.3) can be written as:

$$H^d = -\sum_{i-1}^{L-1} e_i - \sin\gamma\left[i - \tan\left(\frac{\omega-\delta}{2}\right)\right] e_0^d - \sin\gamma\left[i + \tan\left(\frac{\omega+\delta}{2}\right)\right] e_L^d \tag{2.13}$$

We shall see later that the spectral equivalences in the $SU(N)$ chains involve generalizations of (2.9) and (2.13).

2.1 1BTL diagonal representations

At generic points one can construct a similarity transformation between the two Hamiltonians H^{nd} (2.2) and H^d (2.3). This implies that there exists a representation of the 1BTL in the diagonal chain. The result at 2 sites is given by [1]:

$$e_0 = \begin{pmatrix} \frac{\sin\omega}{\sin(\omega+\gamma)} & 0 & 0 & 0 \\ 0 & \frac{\sin\omega}{\sin(\omega+\gamma)} & \frac{\sin\gamma\cos\left(\frac{\omega+\delta}{2}\right)}{\sin(\omega+\gamma)\cos\left(\frac{\omega-\delta}{2}\right)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \eta & \eta\xi & 0 \\ 0 & 1 & \xi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.14}$$

where:

$$\begin{aligned}\eta &= \cos\left(\frac{\delta-2\gamma-\omega}{2}\right)\sec\left(\frac{\delta-\omega}{2}\right)\\ \xi &= \cos\left(\frac{\delta+2\gamma-\omega}{2}\right)\sec\left(\frac{\delta-\omega}{2}\right)\end{aligned} \tag{2.15}$$

Notice that all parameters *including* δ appear in both e_0 and e_1. We shall call this representation 'the *real* diagonal representation' as inserting (2.14) into the integrable Hamiltonian (2.9) we obtain exactly the diagonal Hamiltonian (2.3) for the two site case.

The structure of this representation is difficult to understand due to the additional dependence on δ. However one can use the invertible transformation:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \xi - e^{-i\gamma} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.16}$$

to transform the generators:

$$e_i \to e_i = U e_i U^{-1} \qquad i = 0, 1 \tag{2.17}$$

This preserves S^z and brings the generators into a canonical form:

$$e_0 = \begin{pmatrix} \frac{\sin\omega}{\sin(\omega+\gamma)} & 0 & 0 & 0 \\ 0 & \frac{\sin\omega}{\sin(\omega+\gamma)} & 1-\frac{e^{i\gamma}\sin\omega}{\sin(\omega+\gamma)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e^{i\gamma} & 1 & 0 \\ 0 & 1 & e^{-i\gamma} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.18}$$

In this form e_1 is the standard generator (2.5) and there is no parameter δ in either e_0 or e_1. We shall call this representation 'the *canonical* diagonal representation'. The crucial difference between this representation (2.18) and the 'non-diagonal' representation (2.8) of 1BTL is that e_0 acts now in the spin chain not only on the first site but on *all* the sites of the chain.

The natural framework to understand this canonical diagonal representation is the braid group. This will allow us to generalize it easily to an arbitrary number of sites and also to understand similar structures in all $SU_q(N)$ spin chains. In the next section we shall discuss the relevant braid and boundary algebras that will be required.

In the 1BTL case one can use the loop basis to define another diagonal representation. This is given in appendix A and is the representation which was used in [23].

3 Braid groups and integrability

3.1 Yang-Baxter equation

Integrable systems have an infinite number of conserved charges allowing a large number of properties to be derived exactly. A sufficient condition for integrability is the Yang-Baxter (YB) equation [26]:

$$\begin{aligned} R_i(u)R_{i+1}(u+v)R_i(v) &= R_{i+1}(v)R_i(u+v)R_{i+1}(u) \\ R_i(u)R_j(v) &= R_j(v)R_i(u) \quad |i-j|>1 \end{aligned} \tag{3.1}$$

In this paper we shall be motivated by integrability but will actually mostly work at the level of the braid limits in which the spectral parameters disappear. In this case the YB equation becomes the *braid group* generated by elements g_i with $i = 1, \cdots, L-1$ obeying:

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \\ g_i g_j &= g_j g_i \quad |i-j|>1 \end{aligned} \tag{3.2}$$

An important simplification occurs when each generator only has two eigenvalues. These must be the same for each generator and after a rescaling this amounts to only a single parameter. The resulting algebra, known as the *Hecke algebra*, has relations:

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \qquad i = 1, \cdots, L-1 \\ g_i g_j &= g_j g_i \qquad |i-j|>1 \\ (g_i - 1)(g_i + q^2) &= 0 \end{aligned} \tag{3.3}$$

The great simplification of the Hecke algebra, in contrast to the braid group, comes from its close relation to the symmetric group ($q = 1$ point). We shall set $q = e^{i\gamma}$ as one is mostly interested in cases in which γ is real.

Once one possesses a representation of the Hecke algebra then one can get a solution to the full Yang-Baxter equation using the process of Baxterization (see appendix B.1).

Using the Temperley-Lieb generators (2.5) we can form a representation of the Hecke algebra:

$$g_i = 1 - q e_i \tag{3.4}$$

This is a quotient of the Hecke algebra as these elements also satisfy the identity:

$$g_i g_{i+1} g_i - g_i g_{i+1} - g_{i+1} g_i + g_i + g_{i+1} - \mathbf{1} = 0 \tag{3.5}$$

3.2 Boundary Yang-Baxter equation

In general boundary terms do not preserve the conserved charges of the bulk system and integrability is lost. For the case of a single boundary term at the L.H.S the sufficient condition for integrability to be maintained is the boundary Yang-Baxter, or reflection, equation [27, 28]:

$$\begin{aligned} K_0(v)R_1(u+v)K_0(u)R_1(u-v) &= R_1(u-v)K_0(u)R_1(u+v)K_0(v) \\ R_i(u)K_0(v) &= K_0(v)R_i(u) \quad |i-j|>1 \end{aligned} \tag{3.6}$$

Again one can take a braid limit and, in addition to the braid group relations, we have an extra boundary generator g_0 obeying:

$$\begin{aligned} g_0g_1g_0g_1 &= g_1g_0g_1g_0 \\ g_0g_i &= g_ig_0 \qquad i>1 \end{aligned} \tag{3.7}$$

This is known as the braid group of type B. If the bulk generators also obey the Hecke condition (3.3) then this is called the *affine* Hecke algebra [29]. A further quotient of this system is when g_0 also obeys a quadratic relation:

$$(g_0-1)(g_0-r^2) = 0 \tag{3.8}$$

The resulting algebra is known as the Hecke algebra of type B. Note the slight difference in conventions between the quadratic relations in (3.3) and (3.8). For this quotient a process of Baxterization can again be carried out (see appendix B.2.1) to get a solution to the full reflection equation [16, 17].

The 1BTL algebra (2.7) is a representation of the Hecke algebra of type B (with $r=e^{-i\omega}$). In addition to the Temperley-Lieb braid generators (3.4) we take:

$$g_0 = \mathbf{1} - 2ie^{-i\omega}\sin(\gamma+\omega)e_0 \tag{3.9}$$

This is a quotient of the Hecke algebra of type B as it also obeys the identity:

$$g_1g_0g_1 - g_1g_0 - g_0g_1 + (1+e^{-2i\omega})g_1 + g_0 - (1+e^{-2i\omega})\mathbf{1} = 0 \tag{3.10}$$

The solutions to the reflection equation that will appear in this paper require going beyond the quadratic quotient (3.8) to a cubic one:

$$g_0(g_0-1)(g_0-r^2) = 0 \tag{3.11}$$

with the additional relation:

$$g_1g_0^2g_1g_0^2 - g_0^2g_1g_0^2g_1 = (1+r^2)(g_1g_0^2g_1g_0 - g_0g_1g_0^2g_1) \tag{3.12}$$

We are not aware of previous discussion of this quotient. The Baxterization and related integrable model is discussed in appendix B.2.2. Although the Hecke algebra of type

B is contained within this quotient we have chosen to present it separately as it is an important subset in which many relations simplify dramatically.

It is also possible to add a boundary term to the right end of the chain. In this case the braid limit of the reflection equation is:

$$\begin{aligned} g_L g_{L-1} g_L g_{L-1} &= g_{L-1} g_L g_{L-1} g_L \\ g_L g_i &= g_i g_L \qquad i < L-1 \end{aligned} \tag{3.13}$$

With boundary generators at both ends we have the affine braid group of type B.

In a physical system it is common that the generators are realized locally. For example in a standard spin chain bulk terms g_i are nearest neighbour interactions and the boundary operators g_0 (and g_L) act only on the first (and last) site. However in the abstract formalism of the braid group such a local realization is certainly *not* a requirement.

4 Integrable $SU_q(N)$ spin chains with boundaries

In this section we shall discuss the $SU_q(N)$ spin chains and possible integrable boundary terms that can be added to them.

4.1 Bulk $SU_q(N)$ invariant chain

The bulk generators [30] are given in terms of a local interaction on sites i and $i+1$:

$$g_i = \mathbf{1} \cdots \otimes \mathbf{1} \otimes A \otimes \mathbf{1} \cdots \otimes \mathbf{1} \qquad 1 \leq i \leq L-1 \tag{4.14}$$

where:

$$A = \sum_{n=1}^{N} e_{n,n} \otimes e_{n,n} + q \sum_{n \neq m} e_{n,m} \otimes e_{m,n} + (1-q^2) \sum_{n<m} e_{m,m} e_{n,n} \tag{4.15}$$

and the $e_{m,n}$ are the elementary matrices being only non-zero in the entry of the m'th row and n'th column. It is simple to verify that the g_i obeys the Hecke algebra conditions (3.3). These generators are invariant under the $SU_q(N)$ quantum group symmetry (see section 6).

The corresponding integrable chain is found by Baxterizing this braid matrix to get the R-matrix, then forming the full transfer matrix, and finally extracting the integrable Hamiltonian from this[2]. The result (see appendix B.1) is given by:

$$H = \sum_{i=1}^{L-1} g_i \tag{4.16}$$

[2]When referring to *the* integrable Hamiltonian we shall always mean the simplest one.

There are several different types of integrable boundary terms that can be added to this Hamiltonian. In this paper we shall consider two general classes: diagonal boundary terms added to both ends and a non-diagonal boundary term added to just one end.

4.2 Diagonal boundary terms

We first consider diagonal boundary terms added to the $SU_q(N)$ integrable chain. Here we shall focus on the braid limit as the full solution can be reconstructed by Baxterizing these solutions. The integrable chain with diagonal boundary terms will be given at the end of this subsection.

We consider an arbitrary diagonal matrix K_0^d acting only on the first site:

$$g_0 = K_0^d \otimes \mathbf{1} \cdots \otimes \mathbf{1} \tag{4.17}$$

The general diagonal solution to the reflection equation for $SU_q(N)$ chains was given in [15]. The braid limit of these solutions gives the matrix g_0. All the solutions found for a left boundary g_0^d satisfy (up to rescaling):

$$g_0^d \left(g_0^d - 1 \right) = 0 \tag{4.18}$$

Therefore g_0^d can only have eigenvalues 0, and 1. The different possibilities are distinguished by $Tr(g_0)$ or equivalently the multiplicity of each eigenvalue. We shall denote by ${g_0^d}^{(k)}$ with $k = 1, \cdots, N-1$ the solution with k eigenvalues equal to 1. The general formula for ${K_0^d}^{(k)}$ is:

$${K_0^d}^{(k)} = \sum_{n=1}^{k} e_{n,n} \tag{4.19}$$

We can also add an integrable diagonal term to the right end:

$$g_L = \mathbf{1} \cdots \otimes \mathbf{1} \otimes K_L^d \tag{4.20}$$

Again the generators g_L only have two eigenvalues and the diagonal cases can have (up to rescaling) only eigenvalues 0 and 1. We denote the different solutions by ${g_L^d}^{(k)}$ with $k = 1, \cdots, N-1$ the solution with k eigenvalues equal to 1. The general formula is given by:

$${K_L^d}^{(k)} = \sum_{n=N-k+1}^{N} e_{n,n} \tag{4.21}$$

The first few cases of the left and right diagonal boundary terms are given by:

4.2.1 $SU(2)$

In this case there is only one solution:

- $k = 1$

$$K_0^{d\ (1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad K_L^{d\ (1)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{Eigenvalues :} \quad 0, 1 \qquad (4.22)$$

4.2.2 $SU(3)$

There are now two solutions:

- $k = 1$

$$K_0^{d\ (1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad K_L^{d\ (1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Eigenvalues :} \quad 0, 0, 1 \,(4.23)$$

- $k = 2$

$$K_0^{d\ (2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad K_L^{d\ (2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Eigenvalues :} \quad 0, 1, 1 \,(4.24)$$

4.2.3 $SU(4)$

There are now three solutions:

- $k = 1$

$$K_0^{d\ (1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_L^{d\ (1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{Eigenvalues :} \quad 0, 0, 0, 1$$

- $k = 2$

$$K_0^{d\ (2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_L^{d\ (2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{Eigenvalues :} \quad 0, 0, 1, 1$$

- $k = 3$

$$K_0^{d\ (3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_L^{d\ (3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{Eigenvalues :} \quad 0, 1, 1, 1$$

The integrable chain with diagonal boundary terms on both ends is parameterized by integers (k_1, k_L) where $(1 \leq k_1, k_L \leq N-1)$. It is given by:

$$H = a_0 g_0^{d\ (k_1)} + \sum_{i=1}^{L-1} g_i + a_L g_L^{d\ (k_L)} \tag{4.25}$$

where a_0 and a_L are two arbitrary parameters and the solutions $g_0^{d\ (k)}$ and $g_L^{d\ (k)}$ are those given above.

4.3 Non-diagonal Hecke type boundary

For the non-diagonal cases we shall only examine the case of a single left boundary generator. There are many solutions to the reflection equation related by global gauge transformations and here we shall give the simplest. Indeed one of the main benefits of following an algebraic approach is that it is only sensitive to the real physical parameters of the problem. The addition of a second boundary would restrict the number of possible gauge transformations forcing one to include more physical parameters.

We found that there are two different types of non-diagonal boundary generators which are distinguished by the number of eigenvalues. In this section will shall give the Hecke type ones in which we have have only two eigenvalues. In the next section the non-Hecke ones with three different eigenvalues will be given. We did not find any other solutions.

The solutions lying in the type B Hecke quotient (3.8) have only two eigenvalues: 1 and r^2. The different possibilities are distinguished by $Tr(g_0)$ or equivalently the multiplicity of each eigenvalue. We shall denote by $g_0^{(k)}$ with $k = 1, \cdots, N-1$ the solution with k eigenvalues equal to 1. The boundary generator is given by:

$$g_0^{(k)} = K_0^{(k)} \otimes \mathbf{1} \otimes \mathbf{1} \cdots \otimes \mathbf{1} \tag{4.26}$$

The explicit form is given by:

$$K_0^{(k)} = \sum_{n=1}^{k} e_{n,n} + r^2 \sum_{n=1}^{N-k} e_{n,n} - ir \sum_{n=1}^{min(k,N-k)} (e_{n,N+1-n} + e_{N+1-n,n}) \tag{4.27}$$

These were found by directly solving equation (3.7). The general form of these is not so revealing so let us look at the first few cases:

4.3.1 $SU(2)$

There is only one solution with $k = 1$:

$$K_0^{(1)} = \begin{pmatrix} 1+r^2 & -ir \\ -ir & 0 \end{pmatrix} \quad \text{Eigenvalues}: \quad 1, r^2 \tag{4.28}$$

4.3.2 $SU(3)$

There are now two solutions:

- $k = 1$

$$K_0^{(1)} = \begin{pmatrix} 1+r^2 & 0 & -ir \\ 0 & r^2 & 0 \\ -ir & 0 & 0 \end{pmatrix} \quad \text{Eigenvalues :} \quad 1, r^2, r^2 \tag{4.29}$$

- $k = 2$

$$K_0^{(2)} = \begin{pmatrix} 1+r^2 & 0 & -ir \\ 0 & 1 & 0 \\ -ir & 0 & 0 \end{pmatrix} \quad \text{Eigenvalues :} \quad 1, 1, r^2 \tag{4.30}$$

4.3.3 $SU(4)$

There are now three solutions:

- $k = 1$

$$K_0^{(1)} = \begin{pmatrix} 1+r^2 & 0 & 0 & -ir \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ -ir & 0 & 0 & 0 \end{pmatrix} \quad \text{Eigenvalues :} \quad 1, r^2, r^2, r^2 \tag{4.31}$$

- $k = 2$

$$K_0^{(2)} = \begin{pmatrix} 1+r^2 & 0 & 0 & -ir \\ 0 & 1+r^2 & -ir & 0 \\ 0 & -ir & 0 & 0 \\ -ir & 0 & 0 & 0 \end{pmatrix} \quad \text{Eigenvalues :} \quad 1, 1, r^2, r^2 \tag{4.32}$$

- $k = 3$

$$K_0^{(3)} = \begin{pmatrix} 1+r^2 & 0 & 0 & -ir \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -ir & 0 & 0 & 0 \end{pmatrix} \quad \text{Eigenvalues :} \quad 1, 1, 1, r^2 \tag{4.33}$$

Note that the previous diagonal solutions in section 4.2 are obtained from these by setting $r = 0$.

In this case the most general integrable chain (see appendix B.2.1) with $(1 \leq k < N)$ can be written as:

$$H = -i\frac{e^{i(\gamma+\omega)}\sin\gamma}{\cos\omega+\cos\delta}g_0^{(k)} + \sum_{i=1}^{L-1} g_i \tag{4.34}$$

where δ is arbitrary and we take $q = e^{i\gamma}$ and $r = e^{-i\omega}$. The parameterization is of course arbitrary and we have chosen one for later convenience. Note that as $w \to -i\infty$ (i.e. $r \to 0$) we recover the integrable chain with an arbitrary left diagonal boundary i.e. (4.25) with $a_L = 0$.

4.4 Non-Hecke type boundary

We also found solutions to the reflection equation which did not lie within the Hecke quotient 3.3. These all lie within a cubic quotient:

$$g_0(g_0-1)(g_0-r^2) = 0 \tag{4.35}$$

Therefore g_0 can only have eigenvalues 0, 1, and r^2. The different possibilities are distinguished by $Tr(g_0)$ and $Tr(g_0^2)$ or equivalently the multiplicity of each eigenvalue. We shall denote by $g_0^{(k_1,k_2)}$ the solution with k_1 eigenvalues equal to 1 and k_2 eigenvalues equal to r^2. Clearly we must have $k_1 + k_2 < N$ otherwise we have no zero eigenvalues. We shall see shortly that our solutions also obey an additional relation (4.44). The Baxterization of these solutions is given in appendix B.2.2. For this paper the only purpose of this process is to find the corresponding integrable Hamiltonian. The boundary generator is given by:

$$g_0^{(k_1,k_2)} = K_0^{(k_1,k_2)} \otimes \mathbf{1} \otimes \mathbf{1} \cdots \otimes \mathbf{1} \tag{4.36}$$

The explicit form is given by:

$$K_0^{(k_1,k_2)} = \sum_{n=1}^{k_1} e_{n,n} + r^2 \sum_{n=1}^{k_2} e_{n,n} - ir \sum_{n=1}^{min(k_1,k_2)} (e_{n,k_1+k_2+1-n} + e_{k_1+k_2+1-n,n}) \tag{4.37}$$

These were again found by directly solving equation (3.7). Note that taking $k_1+k_2 = N$, the case in which g_0 has no zero eigenvalues, we reproduce the previous Hecke boundary expression (4.27).

Let us again examine the first few cases:

4.4.1 $SU(2)$

All solutions for $SU(2)$ are of Hecke type (4.28).

4.4.2 $SU(3)$

We have a single solution given by:

$$K_0^{(1,1)} = \begin{pmatrix} 1+r^2 & -ir & 0 \\ -ir & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Eigenvalues :} \quad 0, 1, r^2 \tag{4.38}$$

4.4.3 $SU(4)$

- $k_1 = 1, k_2 = 1$

$$K_0^{(1,1)} = \begin{pmatrix} 1+r^2 & -ir & 0 & 0 \\ -ir & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{Eigenvalues :} \quad 0, 0, 1, r^2 \tag{4.39}$$

- $k_1 = 1, k_2 = 2$

$$K_0^{(1,2)} = \begin{pmatrix} 1+r^2 & 0 & -ir & 0 \\ 0 & r^2 & 0 & 0 \\ -ir & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{Eigenvalues :} \quad 0, 1, r^2, r^2 \tag{4.40}$$

- $k_1 = 2, k_2 = 1$

$$K_0^{(2,1)} = \begin{pmatrix} 1+r^2 & 0 & -ir & 0 \\ 0 & 1 & 0 & 0 \\ -ir & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{Eigenvalues :} \quad 0, 1, 1, r^2 \tag{4.41}$$

Note that these solutions can be obtained by embedding the previous Hecke solutions into a larger matrix. For example in the case $k_1 = 1, k_2 = 2$ we have embedded the matrix (4.29) into a 4×4 matrix. This pattern persists and in this way one obtains all the non-Hecke solutions from the previous Hecke ones. This observation implies that the matrices satisfy additional relations. The combination:

$$X = (1 + r^2)g_0 - g_0^2 \tag{4.42}$$

is just a constant matrix $r^2\mathbf{1}$ for the Hecke type solutions. For the non-Hecke solutions X is equal to $r^2 g_0^{d\ (k_1+k_2)}$ where the $g_0^{d\ (k)}$ is as defined in section 4.2. Therefore as $g_0^{d\ (k)}$ satisfies the reflection equation (3.7) we must have the relation:

$$X g_1 X g_1 \;\; = \;\; g_1 X g_1 X \tag{4.43}$$

Inserting the form of X (4.42) and using the reflection equation (3.7) we find that:

$$g_1 g_0^2 g_1 g_0^2 - (1+r^2) g_1 g_0^2 g_1 g_0 = g_0^2 g_1 g_0^2 g_1 - (1+r^2) g_0 g_1 g_0^2 g_1 \tag{4.44}$$

This additional relation is necessary to Baxterize the solutions. The most general integrable chain ($1 \leq k_1 + k_2 < N$) can be written as (see appendix B.2.2):

$$\begin{aligned} H^{nd} &= -i \frac{e^{-i(\delta-\gamma)} \sin\gamma}{\cos\omega + \cos\delta} \left(1 + e^{i(\omega+\delta)} + e^{2i\omega}\right) g_0^{(k_1,k_2)} && (4.45) \\ &\quad + i \frac{e^{-i(\delta-\gamma)} e^{2i\omega} \sin\gamma}{\cos\omega + \cos\delta} (g_0^{(k_1,k_2)})^2 + \sum_{i-1}^{L-1} g_i && (4.46) \end{aligned}$$

where δ is arbitrary and, as before, we take $q = e^{i\gamma}$ and $r = e^{-i\omega}$.

5 Diagonal representations and the occurrence of spectral equivalences

In this section we shall examine the braid and Hecke algebras with one and two-boundary extensions. We shall show than in certain cases the two boundary algebra contains a one parameter family of one-boundary algebras. This algebraic phenomenon will motivate the construction of a set of spectral equivalences between arbitrary integrable one-boundary and diagonal chains.

5.1 Diagonal representations

Let us begin with the braid group generators (3.2) supplemented by a boundary operator G_0 at the left end and G_L at the right end. We have:

$$g_1 G_0 g_1 G_0 = G_0 g_1 G_0 g_1 \qquad g_{L-1} G_L g_{L-1} G_L = G_L g_{L-1} G_L g_{L-1} \tag{5.47}$$

We can 'braid translate' the operator G_L to the left hand end to produce a non-local generator:

$$G_0^{(NL)} = g_1^{-1} g_2^{-1} \cdots g_{L-1}^{-1} G_L g_{L-1} \cdots g_2 g_1 \tag{5.48}$$

It is simple to verify using only the braid relations that this obeys the reflection equation:

$$g_1 G_0^{(NL)} g_1 G_0^{(NL)} = G_0^{(NL)} g_1 G_0^{(NL)} g_1 \tag{5.49}$$

and commutes with the rest of the bulk generators:

$$\left[g_i, G_0^{(NL)}\right] = 0 \qquad 2 \leq i \leq L-1 \tag{5.50}$$

Now from these two simple solutions G_0 and $G_0^{(NL)}$ we can try to form a one parameter family of solutions to the reflection equation:

$$g_0 = G_0 + r^2 G_0^{(NL)} \tag{5.51}$$

This will obey the reflection equation if:

$$g_1 G_0 g_1 G_0^{(NL)} + g_1 G_0^{(NL)} g_1 G_0 = G_0 g_1 G_0^{(NL)} g_1 + G_0^{(NL)} g_1 G_0 g_1 \tag{5.52}$$

Now it is useful to rewrite $G_0^{(NL)} = g_1^{-1} Y g_1$ where Y commutes with G_0. Then we have:

$$g_1^2 G_0 Y g_1 + g_1 Y g_1^2 G_0 = g_1 G_0 Y g_1^2 + Y g_1^2 G_0 g_1 \tag{5.53}$$

Up to this point we have used purely braid group relations. However if g_1 also obeys the Hecke condition (3.3) then this simplifies to:

$$g_1 Y g_1 G_0 = Y g_1 G_0 g_1 \tag{5.54}$$

which is equivalent to:

$$\left[g_1, G_0^{(NL)} G_0 \right] = 0 \tag{5.55}$$

We shall examine the case in which a stronger condition is obeyed:

$$G_0^{(NL)} G_0 = 0 \tag{5.56}$$

As the bulk generators are invertible the condition (5.57) is equivalent to:

$$G_L g_{L-1} \cdots g_2 g_1 G_0 = 0 \tag{5.57}$$

Therefore if we have integrable boundary terms on the left and right ends obeying (5.57) then we can construct a family of non-local solutions to the reflection equation (3.7). We have assumed throughout that the bulk braid group generators g_i are invertible. However the boundary generators G_0 and G_L cannot be invertible elements otherwise the condition (5.57) reduces to a trivial one.

Let us specialize to the case of interest for this paper in which both the boundary generators are diagonal. In this case they satisfy:

$$G_0(G_0 - 1) = 0 \qquad G_L(G_L - 1) = 0 \tag{5.58}$$

and therefore we have:

$$G_0^{(NL)}(G_0^{(NL)} - 1) = 0 \tag{5.59}$$

Let us now form the powers of the generator g_0 (5.51):

$$g_0^2 = G_0 + r^2 G_0 G_0^{(NL)} + r^4 G_0^{(NL)} \tag{5.60}$$

$$g_0^3 = G_0 + (r^2 + r^4) G_0 G_0^{(NL)} + r^6 G_0^{(NL)} \tag{5.61}$$

We see that we must have the identity:

$$g_0(g_0 - 1)(g_0 - r^2) = 0 \tag{5.62}$$

Now we can form the combination X as in (4.42):

$$X = (1 + r^2)g_0 - g_0^2 = r^2 \left(G_0 + G_0^{(NL)} - G_0 G_0^{(NL)} \right) \tag{5.63}$$

By using the Hecke condition for g_1 (3.3) and the condition (5.57) one can verify, after a great deal of simple but tedious algebra, that algebraically this quantity X also obeys the reflection equation. We did not find a simple way to demonstrate this fact. Therefore the generator g_0 given by (5.51) also obeys the additional relation (4.44).

In the cases in which we have the more restrictive condition:

$$(g_0 - 1)(g_0 - r^2) = 0 \tag{5.64}$$

we have:

$$(G_0 - 1)(G_0^{(NL)} - 1) = 0 \tag{5.65}$$

We have shown that from the two-boundary braid algebra in which the bulk generators obey the Hecke condition and the boundary ones obey (5.57) we can form a one parameter set of non-local solutions to the reflection equation. Conversely if we are given a one-parameter family of solutions, parameterized by r, from (5.51) we can simply read off the individual generators G_0 and $G_0^{(NL)}$. Algebraically these non-local solutions behave in exactly the same way as the non-diagonal solutions of sections 4.3 and 4.4.

It remains for us to discuss *when* the integrable diagonal boundary terms of section 4.2 obey the constraint (5.57). This is the case when $k_1 + k_2 \leq N$. All of the one-boundary non-diagonal terms given in sections 4.3 and 4.4 obey this condition. The more restrictive constraint (5.65) is obeyed when $k_1 + k_2 = N$ which is exactly the point at which the non-Hecke boundary terms reduce to the purely Hecke ones. We shall show in the next subsection that in all the cases in which $k_1 + k_2 \leq N$ there is an exact spectral equivalence between the the diagonal and one-boundary Hamiltonians. We shall return to the question of the diagonal terms which have $k_1 + k_2 > N$ shortly.

5.2 Spectral equivalences

The previous algebraic considerations have shown us that for every set of diagonal generators obeying the condition (5.57) one can form a non-local family of solutions

to the reflection equation. Such solutions are algebraic as they depend only on the parameters entering the algebra and obey the same relations as the general non-diagonal solutions. This is the generalization of the canonical diagonal representation (2.18) from the XXZ case. As explained in section 2 this is distinct from the real diagonal representation (2.14) that also depends on the parameters in the Hamiltonian.

Although we have not managed to understand the structure of the real diagonal representation we were motivated by the existence of the canonical diagonal representation to search for spectral equivalences between one-boundary and diagonal Hamiltonians. These have been verified numerically and by explicit diagonalization at a low number of sites.

5.2.1 Hecke type boundary

There is a spectral equivalence between the following $SU(N)$ integrable chains:

$$H^{Hecke,nd} = -i\frac{e^{i(\gamma+\omega)}\sin\gamma}{\cos\omega+\cos\delta}g_0^{(k)} - \frac{ie^{i(\gamma-\delta)}\sin\gamma}{\cos\omega+\cos\delta} + \sum_{i-1}^{L-1} g_i \tag{5.66}$$

and:

$$\begin{aligned} H^{Hecke,d} &= \sum_{i-1}^{L-1} g_i - e^{i\gamma}\sin\gamma\left[i-\tan\left(\frac{\omega-\delta}{2}\right)\right] g_0^{d\ (k)} \\ &\quad -e^{i\gamma}\sin\gamma\left[i+\tan\left(\frac{\omega+\delta}{2}\right)\right] g_L^{d\ (N-k)} \end{aligned} \tag{5.67}$$

where the non-diagonal Hecke boundary term $g_0^{(k)}$ was given in section 4.3 and the diagonal terms $g_0^{d\ (k)}$ and $g_L^{d\ (N-k)}$ were given in 4.2.

For the $SU(2)$ case the braid operators can be expressed in terms of the 1BTL generators:

$$g_i = \mathbf{1} - e^{i\gamma}e_i \tag{5.68}$$

$$g_0^{(1)} = \mathbf{1} - 2ie^{-i\omega}\sin(\gamma+\omega)e_0 \tag{5.69}$$

$$g_0^d = \frac{1}{2}\left(1+\sigma_1^z\right) \tag{5.70}$$

$$g_L^d = \frac{1}{2}\left(1-\sigma_L^z\right) \tag{5.71}$$

Inserting these definitions into the above equations we reproduce the spectral equivalence found in the XXZ model - see section 2.

5.2.2 Non-Hecke type boundary

There is a spectral equivalence between the following $SU(N)$ integrable chains:

$$H^{Non-Hecke,nd} = -i\frac{e^{-i(\delta-\gamma)}\sin\gamma}{\cos\omega+\cos\delta}\left(1+e^{i(\omega+\delta)}+e^{2i\omega}\right)g_0^{(k_1,k_2)}$$

$$+i\frac{e^{-i(\delta-\gamma)}e^{2i\omega}\sin\gamma}{\cos\omega+\cos\delta}(g_0^{(k_1,k_2)})^2+\sum_{i-1}^{L-1}g_i \tag{5.72}$$

and:

$$\begin{aligned} H^{Non-Hecke,d} &= \sum_{i-1}^{L-1}g_i - e^{i\gamma}\sin\gamma\left[i-\tan\left(\frac{\omega-\delta}{2}\right)\right]g_0^{d\ (k_1)} \\ &\quad -e^{i\gamma}\sin\gamma\left[i+\tan\left(\frac{\omega+\delta}{2}\right)\right]g_L^{d\ (k_2)} \end{aligned} \tag{5.73}$$

where the non-Hecke boundary term $g_0^{(k_1,k_2)}$ was given in section 4.4 and the diagonal terms $g_0^{d\ (k_1)}$ and $g_L^{d\ (k_2)}$ were given in 4.2. We recall from section 4.4 that we always have $k_1+k_2\leq N$. In the extremal case in which $k_2=N-k_1$ the boundary generator g_0 reduces to a Hecke-type boundary and (5.72) becomes the previous Hamiltonian (5.66).

So far we have found spectral equivalences relating all one boundary chains to diagonal ones. However the space of diagonal chains that are involved is limited to the ones satisfying $k_1+k_2\leq N$. One might wonder: what happens to the diagonal chains with $k_1+k_2>N$? As an example consider the following $SU_q(3)$ diagonal boundary terms:

$$K_0^{d\ (2)}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\qquad K_L^{d\ (2)}=\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix} \tag{5.74}$$

These do not obey the constraint (5.57). However let us consider the action of a parity operation, P, on the integrable chain. The bulk generators given in (4.15) transform as:

$$Pg_i(q)=(1-q^2)\mathbf{1}+q^2g_{L-i}(q^{-1}) \tag{5.75}$$

where by $g_{L-i}(q^{-1})$ we mean g_{L-i} with q replaced everywhere by q^{-1}. One can easily verify algebraically that this is an automorphism of the Hecke algebra (3.3). The boundary terms become:

$$Pg_0^{d\ (k_1)}=\mathbf{1}-g_L^{d\ (N-k_1)} \tag{5.76}$$

$$Pg_L^{d\ (k_2)}=\mathbf{1}-g_0^{d\ (N-k_2)} \tag{5.77}$$

In the $SU_q(3)$ example (5.74) we have:

$$P\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\otimes\mathbf{1}\otimes\mathbf{1}\cdots\otimes\mathbf{1}=\mathbf{1}\otimes\mathbf{1}\cdots\otimes\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}=\mathbf{1}-g_L^{d\ (1)} \tag{5.78}$$

$$P\mathbf{1}\otimes\mathbf{1}\cdots\otimes\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\otimes\mathbf{1}\cdots\otimes\mathbf{1}=\mathbf{1}-g_0^{d\ (1)} \tag{5.79}$$

If the original boundary terms $g_0^{d\ (k_1)}$ and $g_L^{d\ (k_2)}$ have $k_1 + k_2 > N$ then it is easy to see that the terms $g_0^{d\ (N-k_2)}$ and $g_L^{d\ (N-k_1)}$ must satisfy $k_1 + k_2 \leq N$. Therefore, up to constant terms, we have transformed our system to one obeying the constraint (5.57). Now all the previous arguments can be used and we again have a spectral equivalence between the diagonal and one-boundary systems. Therefore all possible integrable diagonal boundary terms have spectral equivalences to one boundary chains.

In the next section we shall use the spectral equivalences to understand the patterns of $SU_q(N)$ symmetry breaking that can occur with the addition of an integrable boundary term.

6 $SU_q(N)$ symmetry breaking

The bulk R-matrix (4.15) is well known to be invariant under an $SU_q(N)$ quantum group symmetry [30]. As a consequence the bulk integrable theory also possesses this symmetry. In this section we shall discuss the effect of adding to this a single integrable boundary term.

In the first subsection we shall discuss the bulk symmetry. The different possible integrable boundary terms that can be added to one end have been discussed in sections 4.3 and 4.4. However it is not immediately obvious what symmetries they preserve. We shall use the spectral equivalences of section 5.2 to discuss this problem from the alternative perspective of the diagonal chain where it is much easier to see the symmetries. This reveals a pattern of symmetry breaking patterns by integrable boundary conditions. We shall give explicit examples for $SU(2)$ and $SU(3)$.

Throughout this section we shall work at generic values of the parameters. It is very likely that, as for the XXZ case [1], there can be enhancement of the symmetry for exceptional points but we shall not discuss this here.

6.1 Quantum group symmetry

The $SU_q(N)$ algebra is generated by E^α, F^α and $q^{\pm H^\alpha/2}$ with $\alpha = 1, \cdots, N-1$ subject to the relations:

$$\begin{aligned} q^{H^\alpha/2} E^\beta q^{-H^\alpha/2} &= q^{a_{\alpha\beta}/2} E^\beta \\ q^{H^\alpha/2} F^\beta q^{-H^\alpha/2} &= q^{-a_{\alpha\beta}/2} E^\beta \\ \left[E^\alpha, F^\beta\right] &= \delta_{\alpha\beta} \frac{q^{H^\alpha} - q^{-H^\alpha}}{q - q^{-1}} \end{aligned} \tag{6.80}$$

$$\left[E^\alpha, E^\beta\right] = 0 \quad \left[F^\alpha, F^\beta\right] = 0 \qquad \text{if } a_{\alpha\beta} = 0$$

$$\begin{aligned} E^\alpha E^\alpha E^\beta - (q + q^{-1}) E^\alpha E^\beta E^\alpha + E^\beta E^\alpha E^\alpha &= 0 \qquad \text{if } a_{\alpha\beta} = -1 \\ F^\alpha F^\alpha F^\beta - (q + q^{-1}) F^\alpha F^\beta F^\alpha + F^\beta F^\alpha F^\alpha &= 0 \qquad \text{if } a_{\alpha\beta} = -1 \end{aligned}$$

The numbers $a_{\alpha\beta}$ are the entries of the Cartan matrix A. The co-products are given by:

$$\begin{aligned} \Delta(q^{\pm H^\alpha/2}) &= q^{\pm H^\alpha/2} \otimes q^{\pm H^\alpha/2} \\ \Delta(E^\alpha) &= q^{H^\alpha/2} \otimes E^\alpha + E^\alpha \otimes q^{-H^\alpha/2} \\ \Delta(F^\alpha) &= q^{H^\alpha/2} \otimes F^\alpha + F^\alpha \otimes q^{-H^\alpha/2} \end{aligned} \tag{6.81}$$

Using these we have the following representations on the spin chain:

$$\begin{aligned} q^{\pm H^\alpha/2} &= q^{\pm h^\alpha/2} \otimes \cdots \otimes q^{\pm h^\alpha/2} \\ E^\alpha &= \sum_i q^{h^\alpha/2} \otimes \cdots \otimes q^{h^\alpha/2} \otimes e_i^\alpha \otimes q^{-h^\alpha/2} \otimes \cdots \otimes q^{-h^\alpha/2} \\ F^\alpha &= \sum_i q^{h^\alpha/2} \otimes \cdots \otimes q^{h^\alpha/2} \otimes f_i^\alpha \otimes q^{-h^\alpha/2} \otimes \cdots \otimes q^{-h^\alpha/2} \end{aligned} \tag{6.82}$$

where e^α, f^α, and h^α are the generators in the Chevalley basis of (non-quantum) $SU(N)$. We give below explicit forms in the case of $SU(2)$ and $SU(3)$.

- $SU(2)$

  The Cartan matrix is just a number:

  $$A = 2 \tag{6.83}$$

  There is only one simple root and the generators of $SU(2)$ are given by:

  $$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{6.84}$$

- $SU(3)$ The Cartan matrix is:

  $$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{6.85}$$

  There are now two simple roots and the generators of $SU(3)$ are given by:

  $$h^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad f^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.86}$$

  $$h^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad e^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad f^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{6.87}$$

6.2 Symmetry breaking by integrable boundary terms

The presence of an integrable boundary term will destroy many of the bulk symmetries. We shall only examine the case of a single (left) boundary term. Using the spectral equivalences of the previous section we know we can discuss this problem in the context of the diagonal chain:

$$H = -a_0 g_0^{d\ (k_1)} - \sum_{i=1}^{n} g_i - a_L g_L^{d\ (k_2)} \tag{6.88}$$

where the $g_0^{d\ (k_1)}$ and $g_L^{d\ (k_2)}$ were defined in section 4.2. It is clear, as both boundary terms are diagonal, this Hamiltonian conserves all diagonal charges. However there may also be some additional non-Abelian conserved charges.

In the $SU(2)$ case there is only one integrable boundary term that can be added to the left end. It is given by (4.28):

$$K_0^{(1)} = \begin{pmatrix} 1+r^2 & -ir \\ -ir & 0 \end{pmatrix} \tag{6.89}$$

The spectral equivalence of section 5.2 relates this to a chain with the diagonal boundary terms:

$$K_0^{d\ (1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad K_L^{d\ (1)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.90}$$

The only generators from the quantum group which commutes with these two diagonal terms are q^H and q^{-H}. Therefore $SU_q(2)$ is broken to a single $U(1)$ charge. This is in agreement with the explicit diagonalization [1] of the boundary quantum group charge [11, 12, 14]. We stress that throughout this section we are working at generic values of the parameters where the charges are always fully diagonalizable.

In the $SU(3)$ case there are three distinct integrable boundary terms.

- The first boundary term (4.29) is given by:

$$K_0^{(1)} = \begin{pmatrix} 1+r^2 & 0 & -ir \\ 0 & r^2 & 0 \\ -ir & 0 & 0 \end{pmatrix} \tag{6.91}$$

  In the isospectral diagonal chain this corresponds to the boundary terms:

$$K_0^{d\ (1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad K_L^{d\ (2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.92}$$

  It is obvious that all the diagonal generators commute with these. However in this case there are additional non-Abelian symmetries.

The generator e^2 of the $SU(3)$ group (6.86) commutes with both of these boundary matrices. Now using the co-products (6.81), and the fact that all the diagonal charges are conserved, we see that the generator E^2 commutes with both these boundary generators. A similar argument applies to the generator F^2. Therefore the $SU_q(3)$ quantum group is broken to $SU_q(2) \otimes U(1)$. In the next subsection we shall present numerical results which confirm this pattern of symmetry breaking.

- The second boundary term corresponds to:

$$K_0^{(2)} = \begin{pmatrix} 1+r^2 & 0 & -ir \\ 0 & 1 & 0 \\ -ir & 0 & 0 \end{pmatrix} \tag{6.93}$$

This is isospectral to a chain with diagonal boundary terms:

$$K_0^{d\ (1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad K_L^{d\ (1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.94}$$

This is very similar to the previous one but now, in addition to the diagonal generators, it is E^1 and F^1 that also commute with it. Therefore the $SU_q(3)$ quantum group is again broken to $SU_q(2) \otimes U(1)$.

- The third choice of boundary terms is:

$$K_0^{(1,1)} = \begin{pmatrix} 1+r^2 & -ir & 0 \\ -ir & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.95}$$

This is isospectral to a chain with diagonal boundary terms:

$$K_0^{d\ (1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad K_L^{d\ (1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.96}$$

There are now only the two diagonal generators conserved. Therefore the quantum group is broken to $U(1) \otimes U(1)$.

A similar pattern of symmetry breaking occurs with all the possible integrable boundary terms. For the Hecke boundary terms of section 4.3 with $K_0^{(k)}$ the symmetry breaking is:

$$SU_q(N) \to SU_q(N-k) \otimes SU_q(k) \otimes U(1) \tag{6.97}$$

This symmetry breaking in the diagonal chain was previously described in [31]. For the non-Hecke boundary conditions of section 4.4 with $K_0^{(k_1,k_2)}$ the symmetry breaking is:

$$SU_q(N) \to SU_q(k_1) \otimes SU_q(k_2) \otimes SU_q(N-k_1-k_2) \otimes U(1)^2 \tag{6.98}$$

In both cases the extra $U(1)$ factors are necessary to ensure that we have the full set of diagonal generators.

The case of $SU(2)$, corresponding to the XXZ model, is almost trivial as the symmetry can only break to $U(1)$.

6.3 Numerical example: $SU_q(3)$ symmetry breaking

In this subsection we shall consider a numerical example confirming the previous analysis of symmetry breaking. We consider integrable boundary terms added to the $SU_q(3)$ quantum group invariant Hamiltonian (4.16) on a chain of length $L = 3$. In the first column of the table on page 34 the eigenvalues of the quantum group invariant Hamiltonian with $\gamma = 0.3423$ $(q = e^{i\gamma})$ are given. The degeneracies are as expected from the $SU(3)$ tensor product:

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \tag{6.99}$$

The second column of the table gives the eigenvalues of the Hamiltonian (5.66) with the $g_0^{(1)}$ Hecke boundary term (4.29) and $\omega = 0.54564$, $\delta = -i$. One can see that the quantum group eigenvalues split in the following way:

$$\begin{aligned} \mathbf{1} &\rightarrow \mathbf{1} \\ \mathbf{8} &\rightarrow \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{3} \\ \mathbf{10} &\rightarrow \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{4} \end{aligned} \tag{6.100}$$

These are exactly the branching rules expected from breaking the $SU(3)$ symmetry to an $SU(2)$ subgroup. This is confirmed by repeating the exercise with different lengths of chain. In particular the fundamental representation and its conjugate give:

$$\begin{aligned} \mathbf{3} &\rightarrow \mathbf{1} \oplus \mathbf{2} \\ \bar{\mathbf{3}} &\rightarrow \mathbf{1} \oplus \mathbf{2} \end{aligned} \tag{6.101}$$

The presence of extra Abelian factors in the broken symmetry is not possible to detect by simply observing the spectrum as they do not give rise to degeneracies.

In the third column of the table we have given eigenvalues of the Hamiltonian (5.72) for the case of the $g_0^{(1,1)}$ non-Hecke boundary term (4.38) with again $\omega = 0.54564$, $\delta = -i$. Now one can see that there are no degeneracies consistent with the fact that the quantum group is broken to purely Abelian factors.

7 Conclusion

In this paper we have considered the $SU_q(N)$ model with different types of integrable boundary terms. The single most important result is that the $SU_q(N)$ model with

any integrable non-diagonal boundary term added to one end is iso-spectral to the same $SU_q(N)$ model with purely diagonal boundary terms added to both ends and vice-versa.

In section 2 we reviewed the spectral equivalence found between the XXZ model with diagonal terms on both ends and the same XXZ model with an arbitrary left boundary term [1]. The results at a low number of sites (2.18) motivated the study of the canonical diagonal representation. To understand this structure we required the braid group and Hecke algebras which were given in section 3. In section 4 we introduced the integrable $SU_q(N)$ model and its possible integrable boundary terms. The algebraic description gives one a very transparent way to understand the structure of possible integrable boundary terms. In section 5 we gave the general structure of the canonical diagonal representation. The existence of this suggested spectral equivalences between $SU_q(N)$ models with a single integrable non-diagonal boundary term and those with two diagonal boundaries. Although we have not proved the equivalences of section 5.2 they are consistent with a large number of numerical checks. In section 6 we have used these spectral equivalences to discuss the structure of symmetry breaking that can occur when an integrable boundary term is added to the $SU_q(N)$ model.

There are many outstanding questions raised by this paper. The most urgent is undoubtedly a proof of the spectral equivalences given in section 5.2. In this paper the structure of the real diagonal representation, an example of which is given in (2.14), was not discussed. This appears to be a less algebraic object than the canonical diagonal representation as it also involves the parameters of the Hamiltonian.

The symmetry breaking patterns presented in section 6 are for the case of generic parameters only. The structure of conserved charges in one-boundary systems has been discussed recently in the literature [11, 12, 14] generalizing earlier results at free fermion point [32]. One would expect, as with the XXZ case discussed in [1], that, although the spectral equivalences continue to hold at all points, the diagonal and one-boundary Hamiltonians can have different indecomposable structure. This will occur at the points which the relevant algebra becomes non-semisimple. For the Hecke algebra of type B (3.8) this is known to be when r is a power of q. For the cubic quotient, defined by (3.11) and (3.12), we do not know the corresponding points but it is natural to conjecture that they might be given by the same set. Throughout this paper when we have considered integrable boundaries on both ends we have restricted our consideration to the very special case of two diagonal boundaries. In some cases, but not the XXZ one, one expects part of the quantum group symmetry to survive even for non-diagonal boundary terms at both ends. Finally, it would be extremely interesting to understand extensions of our results to other integrable models.

Acknowledgements

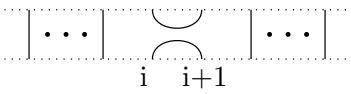
I am grateful for financial support by the E.U. network *Integrable models and applications: from strings to condensed matter* HPRN-CT-2002-00325. I would like to thank V. Rittenberg for many stimulating discussions. I would also like to thank P. Pyatov and A. Ram for discussions on braid and Hecke algebras.

A Link pattern representation of 1BTL

A different representation of the 1BTL algebra, from the spin chain, in another 2^L dimensional vector space is obtained if we use link patterns [24].

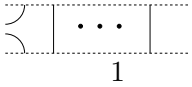
We start by giving the standard graphical representation of the 1 BTL algebra:

e_i =



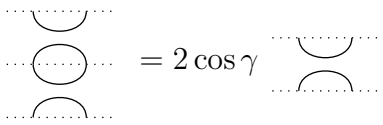
(A.1)

e_0 =

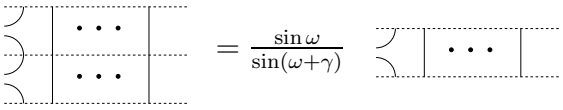


(A.2)

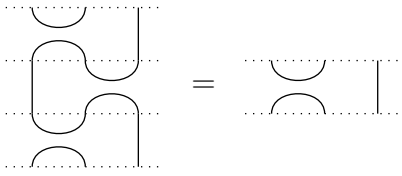
Multiplication of two words in the algebra corresponds to putting one word below the other and merging the loops lines. For example, the relations $e_i^2 = 2\cos\gamma e_i$, $e_0^2 = \frac{\sin\omega}{\sin(\omega+\gamma)} e_0$ and $e_i e_{i+1} e_i = e_i$ graphically read:



(A.3)

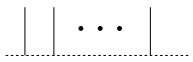


(A.4)

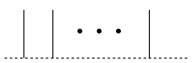


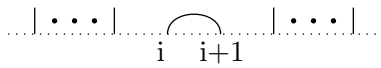
(A.5)

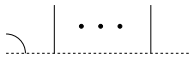
The link pattern representation corresponds to considering an ideal of the 1BTL [24]. We consider the state $|I\rangle$ by taking the graph corresponding to the unit element $\mathbf{1}$ of the 1BTL algebra. It has all L sites unconnected:

$$|I\rangle \quad = \quad$$  (A.6)

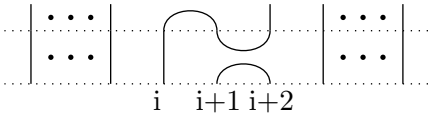
We then act with the algebra and finally keep only the bottom half of the picture:

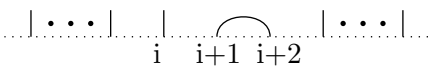
$$\mathbf{1}\,|I\rangle \quad = \quad$$  (A.7)

$$e_i\,|I\rangle \quad = \quad$$ 
 (A.8)

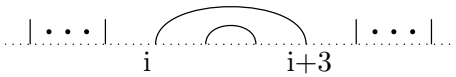
$$e_0\,|I\rangle \quad = \quad$$  (A.9)

Other examples are:

$$e_{i+1}e_i\,|I\rangle \quad = \quad$$ 

$$= \quad$$ 
 (A.10)

and:

$$e_{i+1}e_{i+2}e_i\,|I\rangle \quad = \quad$$ 
 . (A.11)

It is natural to introduce a diagrammatic charge, C, which counts the number of sites which contain loops connected to another site or to the boundary. This diagrammatic charge has *no* algebraic properties and so does not see any particular structure in the 1BTL algebra at the critical points. In general the link patterns form a vector space of dimension 2^L and the diagrammatic charge C splits it into subspaces having the dimension of the binomial coefficients. More specifically for a system of size L we have $C = 0, 1, \cdots, L$ with:

- C even

$$\text{dimension} = \left(\begin{array}{c} L \\ \frac{L}{2} - C \end{array} \right) \tag{A.12}$$

- C odd

$$\text{dimension} = \begin{pmatrix} L \\ -\frac{L}{2} + C - 1 \end{pmatrix} \tag{A.13}$$

However it is important to note that the diagrammatic charge C does *not* commute with the 1BTL generators. However we can define a *new* representation in which it does. First note that, as all elements of the algebra cannot break links (i.e. decrease C), we have a natural decomposition when acting on states of charge c:

$$e_i \mid c\rangle = \tilde{e_i} \mid c\rangle + \{\text{States with C} > \text{c}\} \tag{A.14}$$

It is easy to see that the $\tilde{e_i}$ so defined will obey the 1BTL *and* will commute with C. We call this the chopped diagonal representation.

At a low number of sites we found that this chopped diagonal representation is equivalent at all points to the 1BTL canonical diagonal representation arising in the spin chain.

B Baxterization

In this section we discuss the process of Baxterization [22] for the different boundary conditions that we have given in this paper. The idea is that if one has a solution to the Yang-Baxter or reflection equations in the braid limit (i.e. without spectral parameters) then one may try to reconstruct the full solution with spectral parameters. For the purposes of this paper the form of the full solutions is only required to derive the corresponding integrable chains. For the Yang-Baxter [22] and Hecke type boundary [17] the solutions have previously appeared in the literature and we include them only for completeness. The non-Hecke Baxterization is, to our knowledge, new.

B.1 Yang-Baxter equation

If the bulk generators g_i obey the Hecke algebra (3.3) then using the ansatz:

$$R_i(u) = \mathbf{1} - f(u) g_i \tag{B.1}$$

one finds that $R_i(u)$ obeys the Yang-Baxter equation if:

$$(g_2 - g_1)\Big(f(u) + f(v) + (q^2 - 1) f(u) f(v) - f(u+v)\Big) = 0 \tag{B.2}$$

Let us illustrate how to solve this as all cases proceed in a similar manner. First by putting $u = 0$ we find non-trivial solutions require $f(0) = 0$. Now differentiating w.r.t u and then putting $u = 0$ we get:

$$f'(0) + (e^{2i\gamma} - 1) f(v) f'(0) - f'(v) = 0 \tag{B.3}$$

where $q = e^{i\gamma}$. This is a simple first order equation. The value of $f'(0)$ is simply a scale and with the choice $f'(0) = \frac{2i}{e^{2i\gamma}-1}$ we get the solution:

$$f(u) = \frac{e^{2iu}-1}{e^{2i\gamma}-1} \tag{B.4}$$

The resulting $R_i(u)$ also obeys the so-called unitarity condition:

$$R_i(u)R_i(-u) = \left(1 - \frac{\sin^2 u}{\sin^2 \gamma}\right)\mathbf{1} \tag{B.5}$$

The corresponding integrable chain is given by:

$$\begin{aligned} H &= \sum_{i=1}^{L-1} R_i'(0) \\ &= \frac{2i}{e^{2i\gamma}-1}\sum_{i=1}^{L-1} g_i \end{aligned} \tag{B.6}$$

B.2 Reflection equation

B.2.1 Hecke type boundary

In the case in which g_0 is a generators of the Hecke algebra of type B (3.8) and $R_i(u)$ is given in section (B.1) we make the ansatz:

$$K(u) = \mathbf{1} - a(u)g_0 \tag{B.7}$$

one finds that $K(u)$ obeys the reflection equation if:

$$\begin{aligned} \Big\{e^{2i\omega}f(u+v)\left(a(u)-a(v)\right) + f(u-v)\left(-e^{2i\omega}(1+(-1+q^2)f(u+v)\right)a(v) \\ +a(u)\left((1+e^{2i\omega})a(v) - e^{2i\omega}\right)\Big\}(g_1g_0 - g_0g_1) = 0 \end{aligned} \tag{B.8}$$

This gives:

$$a(u) = \frac{e^{2i\omega}\left(e^{4iu}-1\right)}{1+e^{2i\omega}+e^{i(\omega+\delta+2u)}+e^{i(\omega-\delta+2u)}} \tag{B.9}$$

where the coefficient δ is arbitrary. The corresponding K-matrix (B.7) also satisfies:

$$K(u)K(-u) = \frac{2\left(\cos\delta+\cos(2u-\omega)\right)\left(\cos\delta+\cos(2u+\omega)\right)}{2+\cos 2\delta + 4\cos\delta\cos 2u\cos\omega+\cos 2\omega}\mathbf{1} \tag{B.10}$$

The integrable chain is given by:

$$\begin{aligned} H &= \sum_{i=1}^{L-1} R_i'(0) + \frac{1}{2}K'(0) \\ &= \frac{2i}{e^{2i\gamma}-1}\left\{-\frac{ie^{i(\gamma+\omega)}\sin\gamma}{\cos\omega+\cos\delta}\,g_0 + \sum_{i=1}^{L-1} g_i\right\} \end{aligned} \tag{B.11}$$

B.2.2 Non-Hecke type boundary

In the case in which g_0 satisfies (3.8) and $R_i(u)$ is given in section (B.1) we make the ansatz:

$$K(u) \;=\; \mathbf{1} - a(u)g_0 - b(u)g_0^2 \tag{B.12}$$

Then after a great deal of algebra and using only the reflection equation and Hecke condition for g_1 we find that this leads to:

$$\begin{aligned}(g_1g_0 - g_0g_1)A + (g_1g_0^2 - g_0^2g_1)B + (g_0g_1g_0^2 - g_0^2g_1g_0)C \\ +(g_1g_0^2g_1g_0^2 - g_0^2g_1g_0^2g_1)D = 0\end{aligned} \tag{B.13}$$

where:

$$\begin{aligned}
A &= f(u-v)a(u) - f(u+v)a(u) + f(u-v)a(v) + f(u+v)a(v) \\
&\quad -f(u-v)f(u+v)a(v)(1-q^2) + r^2f(u-v)a(v)b(u) \\
&\quad +r^2f(u-v)a(u)b(v) + r^2(1+r^2)f(u-v)b(u)b(v) \qquad \text{(B.14)} \\
B &= -f(u-v)a(u)a(v) + f(u-v)b(u) - f(u+v)b(u) \\
&\quad -(1+r^2)f(u-v)a(v)b(u) + f(u-v)b(v) + f(u+v)b(v) \\
&\quad -(1-q^2)f(u-v)f(u+v)b(v) - (1+r^2)f(u-v)a(u)b(v) \qquad \text{(B.15)} \\
&\quad -(1+r^2+r^4)f(u-v)b(u)b(v) \\
C &= f(u+v)a(v)b(u) - (1-q^2)f(u+v)f(u-v)a(v)b(u) \\
&\quad -f(u+v)a(u)b(v) \qquad \text{(B.16)} \\
D &= f(u+v)f(u-v)b(u)b(v) \qquad \text{(B.17)}
\end{aligned}$$

Clearly if all the terms in (B.13) were independent we would only have the trivial solution $a(u) = b(u) = 0$. However, as discussed in section 4.4 this is not the case as we have an additional relation:

$$(g_1g_0^2g_1g_0^2 - g_0^2g_1g_0^2g_1) = (1+r^2)(g_1g_0^2g_1g_0 - g_0g_1g_0^2g_1) \tag{B.18}$$

With this extra relation, and using the Hecke condition for g_1 and reflection equation once again, we are able to combine the C and D terms:

$$\begin{aligned}
&(g_0g_1g_0^2 - g_0^2g_1g_0)C + (g_1g_0^2g_1g_0^2 - g_0^2g_1g_0^2g_1)D \\
&\quad = (g_0g_1g_0^2 - g_0^2g_1g_0)\Big\{f(u+v)a(v)b(u) - (1-q^2)f(u+v)f(u-v)a(v)b(u) \\
&\qquad -f(u+v)a(u)b(v) + (1-q^2)(r^2+1)f(u+v)f(u-v)b(u)b(v)\Big\}
\end{aligned} \tag{B.19}$$

In order for (B.12) to satisfy the full reflection equation this must vanish in addition to $A = B = 0$. Using the solution for $f(u)$ and solving first $A = 0$ and $B = 0$ we get

a solution for $a(u)$ and $b(u)$ with arbitrary coefficients $a_1 = a'(0)$ and $b_1 = b'(0)$. Now the vanishing of equation (B.19) requires an additional relation between a_1 and b_1:

$$\Big(a_1 + b_1(1+r^2)\Big)\Big(a_1^2 + r^2 b_1^2 - 4ib_1 + (r^2+1)a_1 b_1\Big) = 0 \tag{B.20}$$

The solution $a_1 = -(1+r^2)b_1$ is not new as we have $(1+r^2)g_0 - g_0^2 = g_0^d$ and so it is just a diagonal solution. The other two quadratic solutions are new. They can be parameterized by:

$$a_1 \;=\; \frac{2ie^{i\omega}}{\cos\omega + \cos\delta} \qquad b_1 = \frac{2ie^{-i(\pm\delta - 2\omega)}}{\cos\omega + \cos\delta} \tag{B.21}$$

where δ is arbitrary. Clearly these are trivially related and we shall choose only the $(+)$ sign. This leads to the following solution for $a(u)$ and $b(u)$:

$$a(u) = \frac{e^{2i\omega}\left(e^{4iu}-1\right)\left(e^{2iu} - 1 - e^{2i\omega} + e^{2i(\omega+u)} + e^{i(\delta+\omega+2u)}\right)}{\left(e^{2iu} - 1 + e^{2i(u+\omega)} + e^{i(\delta+\omega+2u)}\right)\left(e^{2iu} - e^{2i\omega} + e^{2i(\omega+u)} + e^{i(\delta+\omega+2u)}\right)} \tag{B.22}$$

$$b(u) = \frac{e^{4i\omega}\left(e^{4iu}-1\right)}{\left(e^{2iu} - 1 + e^{2i(u+\omega)} + e^{i(\delta+\omega+2u)}\right)\left(e^{2iu} - e^{2i\omega} + e^{2i(\omega+u)} + e^{i(\delta+\omega+2u)}\right)} \tag{B.23}$$

The corresponding K-matrix (B.12) also satisfies:

$$K(u)K(-u) = \mathbf{1} \tag{B.24}$$

Finally we find the integrable chain is given by:

$$\begin{aligned} H \;&=\; \sum_{i=1}^{L-1} R_i'(0) + \frac{1}{2}K'(0) \\ &=\; \frac{2i}{e^{2i\gamma}-1}\Bigg\{-i\frac{e^{-i(\delta-\gamma)}\sin\gamma}{\cos\omega+\cos\delta}\left(1 + e^{i(\omega+\delta)} + e^{2i\omega}\right) g_0^{(k_1,k_2)} \\ &\qquad +i\frac{e^{-i(\delta-\gamma)}e^{2i\omega}\sin\gamma}{\cos\omega+\cos\delta}(g_0^{(k_1,k_2)})^2 + \sum_{i-1}^{L-1} g_i\Bigg\} \end{aligned} \tag{B.25}$$

After this derivation two facts should be emphasized. Firstly the additional quotient (B.18) was essential in order to carry out the Baxterization and obtain the full $K(u)$ matrix. Secondly, once g_i and g_0 are fixed, the integrable Hamiltonian has only *one* free parameter δ.

References

[1] A. Nichols, V. Rittenberg, and J. de Gier, J. Stat. Mech. , P03003 (2005), cond-mat/0411512.

[2] F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter, and G. R. W. Quispel, J. Phys. **A20**, 6397 (1987).

[3] F. C. Alcaraz, M. N. Barber, and M. T. Batchelor, Ann. Phys. **182**, 280 (1988).

[4] V. Pasquier and H. Saleur, Nucl. Phys. **B330**, 523 (1990).

[5] H. N. V. Temperley and E. Lieb, Proc. Roy. Soc. London **A322**, 251 (1971).

[6] P. P. Martin, *Potts Models and Related Problems in Statistical Mechanics* (World Scientific, 1991).

[7] P. Martin and H. Saleur, Commun. Math. Phys. **158**, 155 (1993), hep-th/9208061.

[8] P. Martin and H. Saleur, Lett. Math. Phys. **30**, 189 (1994), hep-th/9302094.

[9] P. P. Martin and D. Woodcock, Journal of Algebra **225**, 957 (2000).

[10] P. P. Martin and D. Woodcock, LMS J. Comput. Math. **6**, 249 (2003), math.RT/0205263.

[11] G. W. Delius and N. J. MacKay, Commun. Math. Phys. **233**, 173 (2003), hep-th/0112023.

[12] G. W. Delius and A. George, Quantum group symmetry of integrable models on the half- line, in *Workshop On Integrable Theories, Solitons And Duality (Sao Paulo, Brazil 2002)*, JHEP, 2002, hep-th/0212300.

[13] R. I. Nepomechie, Lett. Math. Phys. **62**, 83 (2002), hep-th/0204181.

[14] A. Doikou, (2004), math-ph/0402067.

[15] H. J. de Vega and A. Gonzalez Ruiz, J. Phys. **A26**, L519 (1993), hep-th/9211114.

[16] D. Levy and P. Martin, J. Phys. **A27**, L521 (1994).

[17] A. Doikou and P. P. Martin, J. Phys. **A36**, 2203 (2003), hep-th/0206076.

[18] A. Doikou, (2004), math-ph/0409060.

[19] J. Abad and M. Rios, Phys. Lett. **B352**, 92 (1995), hep-th/9502129.

[20] A. Lima-Santos, Nucl. Phys. **B644**, 568 (2002), nlin/0207028.

[21] W.-L. Yang and Y.-Z. Zhang, JHEP **12**, 019 (2004), hep-th/0411160.

[22] V. F. R. Jones, Int. J. Mod. Phys. **A6**, 2035 (1991).

[23] J. de Gier and P. Pyatov, J. Stat. Mech. , P03002 (2004), hep-th/0312235.

[24] J. de Gier, Loops, matchings and alternating-sign matrices, in *14th International Conference on Formal Power Series and Algebraic Combinatorics (Melbourne 2002)*, 2002, math.CO/0211285.

[25] J. de Gier, A. Nichols, P. Pyatov, and V. Rittenberg, (2005), hep-th/0505062, NPB to appear.

[26] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press Inc.(London) Ltd, 1982).

[27] I. V. Cherednik, Theor. Math. Phys. **61**, 977 (1984).

[28] E. K. Sklyanin, J. Phys. **A21**, 2375 (1988).

[29] A. Ram and J. Ramagge, Affine hecke algebras, cyclotomic hecke algebras and clifford theory, in *A tribute to C.S. Seshadri: Perspectives in Geometry and Representation theory*, edited by V. Lakshmibai *et al.*, pp. 428–466, Hindustan Book Agency, New Delhi, 2003, math.RT/0401322.

[30] M. Jimbo, Int. J. Mod. Phys. **A4**, 3759 (1989).

[31] A. Doikou and R. I. Nepomechie, Nucl. Phys. **B530**, 641 (1998), hep-th/9807065.

[32] L. Mezincescu and R. I. Nepomechie, Int. J. Mod. Phys. **A13**, 2747 (1998), hep-th/9709078.

	Integrable boundary term	
Quantum Group	Hecke	Non-Hecke
-1.5493 - 1.2647 i	-1.5728 - 1.4548 i	-1.5752 - 1.3903 i
-0.7167 - 0.9680 i	-0.7277 - 1.1536 i	-0.7345 - 1.1230 i
-0.7167 - 0.9680 i	-0.7277 - 1.1536 i	-0.7276 - 0.9993 i
-0.7167 - 0.9680 i	-0.7189 - 1.1505 i	-0.7189 - 1.1505 i
-0.7167 - 0.9680 i	-0.7189 - 1.1505 i	-0.7026 - 0.9961 i
-0.7167 - 0.9680 i	-0.7189 - 1.1505 i	-0.6767 - 1.0531 i
-0.7167 - 0.9680 i	-0.6283 - 1.1182 i	-0.6463 - 1.0907 i
-0.7167 - 0.9680 i	-0.6283 - 1.1182 i	-0.6283 - 1.1182 i
-0.7167 - 0.9680 i	-0.6159 - 1.1138 i	-0.6190 - 1.0876 i
1.1673 - 0.2967 i	1.1563 - 0.4823 i	1.1315 - 0.3870 i
1.1673 - 0.2967 i	1.1563 - 0.4823 i	1.1331 - 0.3931 i
1.1673 - 0.2967 i	1.1814 - 0.4734 i	1.1814 - 0.4734 i
1.1673 - 0.2967 i	1.1814 - 0.4734 i	1.1833 - 0.4727 i
1.1673 - 0.2967 i	1.1833 - 0.4727 i	1.1902 - 0.3879 i
1.1673 - 0.2967 i	1.1833 - 0.4727 i	1.1904 - 0.4362 i
1.1673 - 0.2967 i	1.1833 - 0.4727 i	1.2057 - 0.3805 i
1.1673 - 0.2967 i	1.1939 - 0.4689 i	1.2084 - 0.3863 i
2.0000 + 0.0000 i	1.9890 - 0.1856 i	1.9890 - 0.1856 i
2.0000 + 0.0000 i	1.9890 - 0.1856 i	2.0000 + 0.0000 i
2.0000 + 0.0000 i	1.9890 - 0.1856 i	2.0317 - 0.1260 i
2.0000 + 0.0000 i	1.9890 - 0.1856 i	2.0342 - 0.0579 i
2.0000 + 0.0000 i	2.0901 - 0.1496 i	2.0707 - 0.0192 i
2.0000 + 0.0000 i	2.0901 - 0.1496 i	2.0901 - 0.1496 i
2.0000 + 0.0000 i	2.0901 - 0.1496 i	2.0994 - 0.0809 i
2.0000 + 0.0000 i	2.1259 - 0.1368 i	2.1157 - 0.0704 i
2.0000 + 0.0000 i	2.1383 - 0.1324 i	2.1259 - 0.1368 i
2.0000 + 0.0000 i	2.1383 - 0.1324 i	2.1383 - 0.1324 i

Table 1: Eigenvalues of an $SU_q(3)$ Hamiltonian at $L = 3$ sites with Hecke and non-Hecke integrable boundary terms. The values of parameters used are given in subsection 6.3.